ON THE SOLUTION OF SOME BOUNDARY-CONTACT PROBLEMS OF LINEAR HYDRODYNAMICS

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We shall consider the formulation of problems of wave propagation in the presence of boundaries on which high-order homogeneous boundary conditions are satisfied. It will turn out that on contours of discontinuities in the coefficients we must give additional conditions, in this paper called "contact conditions". A simple problem will be solved with the application of contact conditions – the reflection of a transverse wave from the lines of discontinuity of an elastic plate, covering a liquid half-space.

Recently, some attention has been given to problems of wave propagation in the presence of surfaces on which homogeneous boundary conditions are given that include derivatives of higher order. Thus, in particular, is described the impact of a sound wave on an elastic shell subjected to pure bending, or on a thin membrane. Lamb [1] has considered the diffraction of a wave by a semi-infinite plate.

Also interesting is the case when the coefficients of the homogeneous boundary condition are piecewise constant, corresponding, for example, to a shell composed of several homogeneous pieces. These problems possess some specific features, and questions of their being properly posed are answered by the uniqueness theorem, given below.

Let a uniform liquid (with density σ and sound speed c) occupy some volume V, bounded by a surface S (which, for example, may be considered to belong to the class BH in the sense of Liapunov); parts of S, in general, may extend to infinity. Let a sound field (with velocity potential ϕ, pressure p, velocity $\mathbf{v}$) satisfy the following:

1) In the volume V, the wave equation holds:

$$\triangle\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = -f(\mathbf{r}, t) \tag{1}$$

where $f(r, t)$ is the source function, differing from zero only in the finite part of V.

2) Zero initial condition holds:

$$P = -\rho \frac{\partial \varphi}{\partial t} = 0, \quad \mathbf{v} = \operatorname{grad} \varphi = 0 \quad \text{for} \quad t = 0 \tag{2}$$

3) The following boundary condition holds on the finite part of surface S (which is considered as partitioned into several pieces S_k):

$$D_k \nabla^4 u - \alpha_k \nabla^2 u + \beta_k u + \mu_k \frac{\partial^2 u}{\partial t^2} = -\rho \frac{\partial \varphi}{\partial t}, \qquad \frac{\partial u}{\partial t} = \frac{\partial \varphi}{\partial n} \quad \text{on } S_k \tag{3}$$

Here the parameters D_k, α_k, β_k and μ_k are constant and positive on each S_k, and u is a function of time and of the points on the surface; eliminating it from (3), we may obtain the homogeneous boundary condition in the following form:

$$\frac{\partial}{\partial n} \left\{ D_k \nabla^4 \varphi - \alpha_k \nabla^2 \varphi + \beta_k \varphi + \mu_k \frac{\partial^2 \varphi}{\partial t^2} \right\} = -\rho \frac{\partial^2 \varphi}{\partial t^2} \quad \text{on } S_k \tag{4}$$

Here $\mathbf{n}$ is the outward normal unit vector, and ∇^k designate differential operators in coordinates normal to $\mathbf{n}$.

4) On the contours of discontinuity of parameters L_k some set of linear conditions holds, insuring the continuity of the normal components of the following vector $\mathbf{I}$ (which is defined on the surface S):

$$\mathbf{I} = D_k \left(\nabla^3 u \frac{\partial u}{\partial t} - \nabla^2 u \frac{\partial \nabla u}{\partial t} \right) - \alpha_k \nabla u \frac{\partial u}{\partial t} \tag{5}$$

(This condition will be called contact condition in what follows). For conditions (1) to (5) the solution to the posed boundary-value problem in linear acoustics will be unique (if one excludes from consideration the uninteresting class of functions which differ from zero on a set of measure zero).

Before proving the statement we make some observations on some of the previous conditions. Equation (3) has a simple meaning, relating force to displacement, if we identify u as the normal displacement of the shell from equilibrium position, $D_k \nabla^4 u$ as the force due to bending, and $\alpha_k \nabla^2 u$ as the force of the membrane type, characteristic of capillary phenomena. The term $\beta_k u$ is interpreted as the quasi-elastic force, μ_k the mass per unit area of the shell, and $\mu_k \partial^2 u/\partial t^2$ the corresponding inertia force. The positive sign of the constants D_k, α_k, μ_k and β_k corresponds to the stability condition for the boundary: the forces resulting from its deformation tend to restore it to equilibrium position ($u = 0$).

The contact condition also admits a simple interpretation. The vector

I is the energy-flow density vector for the shell considered; the first term in (5) is connected with the transfer of elastic energy for bending vibrations (as was shown in our paper [2]), and the second with the energy transferred along the membrane. Higher-order boundary conditions correspond to (physically) rather complicated surfaces, which appear able to transmit energy independently; from this point of view, contact conditions are completely legitimate.

To prove the uniqueness theorem, we shall construct the difference of two possible solutions ($p = p_1 - p_2$, $\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2$, $\phi = \phi_1 - \phi_2$). The following energy theorem is valid [3]:

$$\frac{\partial}{\partial t}\int\limits_V \left(\rho\frac{v^2}{2} + \frac{p^2}{2\rho c^2}\right) dV + \oint\limits_S p v_n \, dS = 0 \tag{6}$$

where the second integral extends only to a finite part of the surface S (since we are considering the unsteady problem). Using boundary condition (3), and carrying out the obvious integration by parts, we may write for some part of the surface S_k

$$\int\limits_{S_k} p v_n \, dS = \oint\limits_{L_k} I_t \, dl + \frac{\partial}{\partial t}\frac{1}{2}\int\limits_{S_k}\left[D_k(\nabla^2 u)^2 + \alpha_k(\nabla u)^2 + \beta_k u^2 + \mu_k\left(\frac{\partial u}{\partial t}\right)^2\right] dS \tag{7}$$

Here I_t is the component of vector I normal to the boundary L_k, I being determined in (5); t is the outward normal.

Into Formula (6) enters the sum over all regions S_k; thus the integral for each contour L_k appears twice, with opposite signs of the unit normal vector. The contact conditions are assumed linear, and thus I_t is continuous across a non-uniform field, so that

$$\sum_k \oint\limits_{L_k} I_t \, dl = 0$$

and the energy theorem takes the form

$$\frac{\partial}{\partial t}\frac{1}{2}\left\{\int\limits_L \left(\rho v^2 + \frac{p^2}{\rho c^2}\right) dV + \sum_k \int\limits_{S_k}\left[D_k(\nabla^2 u)^2 + \alpha_k(\nabla u)^2 + \beta_k u^2 + \mu_k\left(\frac{\partial u}{\partial t}\right)^2\right] dS\right\} = 0 \tag{8}$$

By virtue of the zero initial conditions for each section and the non-negativeness of the coefficients (D_k, α_k, β_k, μ_k), it follows from (8) that the difference of the solutions is identically zero.

Thus, we have proved the uniqueness theorem for the problem with initial data, and shown the necessity to take into account contact conditions. Evidently, they are also needed in more general cases. To formulate the theorem for stationary problems, we replace condition (2) by the radiation principle (in appropriate form). If the region V contains points, where the solution may lose analyticity (edge of a wedge or, in our case, the boundary of the contact surface), then it is necessary to impose conditions of weak singularities for the field, i.e. continuity of the potential ϕ. These variations of the uniqueness theorem have been discussed (cf., for example, [4]).

We now consider a simplest boundary-contact problem.

Let the half-space $y < 0$ be filled with an incompressible fluid of density σ, covered with a thin plate of constant rigidity D, the mass of which is negligible (Fig. 1). On the line ($x = 0$, $y = 0$) the contact condition is disturbed. For instance, both edges of the half-plates may be free, or may be hinged together; the presence of welding would correspond to the trivial case of an infinite homogeneous plate. If a surface-bending wave hits this contact line, it is interesting to determine the strength of the reflected wave, since this problem is connected with the reflection of bending waves from cracks in ice sheets; the hinged-joint case may give some representation on the reflection of bending waves from overlapping ice-sheets. As follows from [2], there exists a range of frequencies for this the proposed model is acceptable.

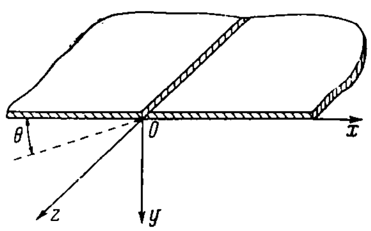

Fig. 1.

The proposed problem will now be reduced to the uniqueness theorem. The velocity potential in the fluid satisfies the Laplace equation

$$\Delta\varphi = 0 \tag{9}$$

and, considering a single frequency, we have for everywhere on the plate (except the contact line) the boundary condition (4)

$$\frac{d}{dy}\nabla^4\varphi - q^5\varphi = 0 \text{ for } \begin{cases} y = 0 \\ x \neq 0 \end{cases} \left(q = \left[\frac{\omega^2\rho}{D}\right]^{1/5}\right) \tag{10}$$

Here the operator ∇ acts only along coordinates x and z, and q designates the wave number for the bending wave on the uniform plate.

The contact conditions on the crack, by virtue of the arbitrariness of the displacement u and the slope $\partial u/\partial x$ of the edges of the half-plate, are given in the form

$$\lim \nabla^2 u = 0 \quad \text{for } x \to \pm 0, \qquad \lim \frac{\partial\nabla^2 u}{\partial x} = 0 \quad \text{for } x \to \pm 0 \tag{11}$$

These conditions coincide with the well-known "free-edge" conditions in the theory of plates. We observe that in the case of the hinged joint (displacement continuous but slopes arbitrary) the contact conditions are formulated somewhat differently:

$$\lim_{x \to \pm 0} \nabla^2 u = 0, \qquad \lim_{x \to +0} u = \lim_{x \to -0} u$$
$$\lim_{x \to +0} \frac{\partial \nabla^2 u}{\partial x} = \lim_{x \to -0} \frac{\partial \nabla^2 u}{\partial x} \tag{12}$$

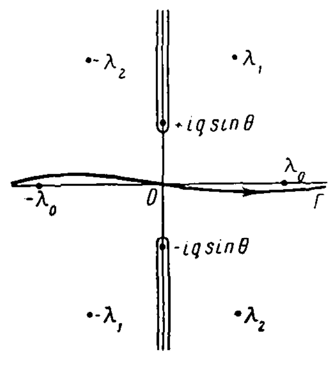

Fig. 2.

Condition (11), because of the second equation of (3), is easily transformed into a condition on the velocity potential ϕ.

The requirement of weak singularities (absence of sources) indicates the continuity of the potential on the line ($x = 0$, $y = 0$).

The radiation principle is easily applied in the following manner. We exclude from the full field ϕ a wave striking against and bending the plane

$$\varphi_0 = Ae^{iq\,(x \cos\theta + z \sin\theta) - qy} \tag{13}$$

where θ is the incidence angle of the wave on the line of contact; then the secondary field $\phi_1 = \phi - \phi_0$ arising on account of the fracture, will be an outgoing wave as $|x| \to \infty$ and will vanish for $y \to \infty$.

We shall seek the "secondary" field ϕ_1 in the form $\Phi(x, y)$ exp iq sin θz where $\Phi(x, y)$ is represented by the following contour integral:

$$\Phi\,(x, y) = \frac{1}{2\pi i} \int_\Gamma f\,(\lambda) \exp\,[i\lambda x - \sqrt{\lambda^2 + q^2 \sin^2\theta}\, y]\, d\lambda \tag{14}$$

Here $f(\lambda)$ is a function to be determined; contour Γ is selected in the λ-plane, as shown in Fig. 2, and the branch of the root is fixed so that on our sheet of the Riemann surface, Re $\sqrt{(\lambda^2 + q^2\sin^2\theta)} \geqslant 0$; this corresponds to the requirement for a vanishing field for $y \to \infty$.

The continuity of the potential will be guaranteed if we assume that $f(\lambda)$ vanishes at infinity faster than $|\lambda|^{-1}$. Accepting that (justification to follow), it is easily verified that, for $x > 0$, the integral in (14) may be reduced to an integral on the contour Γ_+, enclosing all the singularities of the integrand lying above the contour Γ (in the upper half-plane of λ), and for $x < 0$, the initial contour Γ is transformed into the contour Γ_-, enclosing all singularities located below Γ. The integrals on the contours Γ_+ and Γ_- (as will be evident in what follows) admit at least five derivatives in the coordinates x and y. Taking this into account, one readily sees that boundary condition (10), imposed on the secondary field, reduces to the following relation for the unknown function $f(\lambda)$:

$$\frac{1}{2\pi i}\int\limits_{\Gamma_\pm} f(\lambda)\left([\lambda^2+q^2\sin^2\theta]^{5/2}-q^5\right)e^{i\lambda x}\,d\lambda=0 \tag{15}$$

Here

$$\Gamma_+ \text{ is for } x>0, \qquad \Gamma_- \text{ for } x<0$$

By virtue of the arbitrary nature of x, (15) will hold if the function

$$F(\lambda)=f(\lambda)\left[(\lambda^2+q^2\sin^2\theta)^{5/2}-q^5\right]$$

possesses no singularities in the entire complex plane, except at infinity, where it may have a pole of finite order, i.e. if $F(\lambda)$ is some polynomial in λ. The requirement of the continuity of the potential on the line of contact in the crack problem implies that $F(\lambda)$ contains no terms higher than the third power, then $f(\lambda)\sim 1/\lambda^2$ as $\lambda\to\infty$ and integral (14) is continuous. We observe that in the hinged-joint problem the condition of continuity of displacement, i.e. $\partial\phi/\partial y$, demands that $F(\lambda)$ be a second-order polynomial. And so, if

$$f(\lambda)=\frac{c_0+c_1\lambda+c_2\lambda^2+c_3\lambda^3}{\sqrt{(q^2\sin^2\theta+\lambda^2)^5}-q^5} \tag{16}$$

then boundary condition (10) obtains. The explicit form of $f(\lambda)$ indicates that the integrand in (14) has singularities in the form of branch points and poles; on our sheet of the Riemann surface there are six poles, whose positions are shown schematically in Fig. 2. It is also evident that the choice of the contour Γ corresponds to the radiation principle.

It remains to impose the contact conditions on the full field; for the case of the crack problem there are eight such conditions – according to the number of undetermined real constants in (16). Evidently, by virtue of the linearity of relation (11), to determine $c_k = a_k + ib_k$, one will obtain a system of eight linear nonhomogeneous algebraic equations, proportional to the amplitude of the incident wave A. The coefficients of the system are determined from the residues of the poles of $f(\lambda)$ and the integrals along the boundaries of the discontinuities. Considering c_k to be known in principle, it is easy to investigate the obtained solution. Thus the secondary wave consists of an undamped bending wave emanating from the crack (described by the residues of the points $\pm\lambda_0$), a non-uniform bending wave whose amplitude decreases exponentially with increased $|x|$ (described by the residues of the points $\pm\lambda_1$ and

$\pm \lambda_2$), and a three-dimensional disturbance going downward into the fluid whose amplitude decreases as some inverse power of distance.* This disturbance is described by the integral along the boundaries of the discontinuity.

Particularly simple is the case of the normal-incidence ($\theta = 0$) bending wave on the crack; here the computation can be carried to the end without using numerical analysis. The positions of the poles for $\theta = 0$ are evidently

$$\lambda_0 = q, \qquad \lambda_1 = q \exp \frac{i2\pi}{5}, \qquad \lambda_2 = q \exp \frac{-i2\pi}{5}$$

We write one of the contact conditions (11) explicitly:

$$\operatorname{Re} \lim \frac{\partial^3 \varphi}{\partial x^2 \partial y} = \operatorname{Re} \left\{ \frac{F(\lambda_0)}{5\lambda_0} + \frac{F(\lambda_1)}{5\lambda_1} - \frac{F(-\lambda_2)}{5\lambda_2} \right\} + \operatorname{Re} \frac{q^5}{\pi i} \int_0^{i\infty} \frac{\lambda^3 F(\lambda)}{\lambda^{10} - q^{10}} d\lambda + Aq^3 = 0 \tag{17}$$

The real part of the integral in (17) is easily computed with the help of the following device:

$$\operatorname{Re} I_k = \operatorname{Re} \frac{q^5}{\pi i} \int_0^{i\infty} \frac{c_k \lambda^{k+3}}{\lambda^{10} - q^{10}} d\lambda = \operatorname{Re} \frac{q^5 \rho_k}{\pi i} e^{i\varepsilon_k} \int_0^{i\infty} \frac{\lambda^{k+3} d\lambda}{\lambda^{10} - q^{10}}$$

where the real quantities ρ_k and ϵ_k have the following condition imposed on them:

$$\rho_k \cos \varepsilon_k = a_k \qquad (k \text{ even}), \qquad \rho_k \sin \varepsilon_k = b_k \qquad (k \text{ odd})$$

If now ϵ_k is fixed such that

$$\varepsilon_k + (4 + k) \frac{2\pi}{5} = \frac{\pi}{2}$$

then evidently

$$\operatorname{Re} \left\{ \frac{q\rho_k}{\pi i} e^{i\varepsilon_k} vp \int_0^{\infty \exp(i2\pi/5)} \frac{\lambda^{k+3} d\lambda}{\lambda^{10} - q^{10}} \right\} = 0$$

The last integral is taken along the ray $\arg \lambda = 2\pi/5$, on which lies the pole λ_1 of the integrand; thus to deform the initial integration path to the chosen one, we must add (with minus sign) the half-residue at λ_1. As a result, we obtain

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* In the case of the normal-incidence wave, the decrease is as r^{-3}.

$$\operatorname{Re} I_k = -\frac{a_k q^{k-1}}{10\cos\varepsilon_k} \quad (k \text{ even}), \quad \operatorname{Re} I_k = -\frac{b_k q^{k-1}}{10\sin\varepsilon_k} \quad (k \text{ odd})$$

The remaining contact conditions are very similar to (17); this not only permits analogous methods to be applied for the determination of coefficients, but also the simplification of the system of equations with the help of a series of identity transformations. Its solution has the form.

$$c_0 = c_1 = 0, \qquad c_2 = (-1.8 + i2.4)\, Aq^2, \qquad c_3 = (-4.5 - i1.5)\, Aq^3 \tag{18}$$

For the case of the hinge-joint we similarly obtain

$$c_0 = c_1 = 0, \qquad c_2 = (-1.8 + i2.4)\, Aq^2 \tag{19}$$

From (18) and (19) we may deduce the following values for the coefficients of the reflected wave V and the passed wave W for normal incidence: $V = 0.95$ and $W = 0.32$ for the case of a crack; $V = 0.6$ and $W = 0.8$ for the case of a hinged joint. For an oblique-incidence wave the reflection must increase, and the order of the effect is clear from the conclusions presented above. We observe that the wave reflected from the crack is strong – only 10 per cent of the energy of the wave passes through. This fact, that the quantities V and W turn out to be frequency-independent, corresponds to the existence of only one parameter q in the problem. Under actual conditions for sufficiently low frequencies, bending waves on the surface of a frozen sea transform into gravity bending waves and pure gravity waves. The conclusion that the reflection coefficient is constant in this frequency range, naturally, is inapplicable.

In conclusion, we may indicate that by similar means we can obtain the solution for the reflection of wave fronts on a discontinuity line for a membrane. In this case $V = \sqrt{3}/2$, and $W = 1/2$.

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